Recursion - 5

29 October 2020 09:58

Non Homo. Linear Recc Relation with const coeffs

$$f(n) + c_1 f(n-1) + c_2 f(n-2) + \dots + c_k f(n-k) = q(n)$$

where $q(n) \neq 0$ and c_i's are constant.

Solution of Non Homo. Linear Recc. Relation with const coeffs

- Homo solution $f^h(n)$
- Particular solution $f^p(n)$

$\therefore$ General solution $f(n) = f^h(n) + f^p(n)$

Method to find Particular solution :→

Let $f(n) + c_1 f(n-1) + c_2 f(n-2) + \dots + c_k f(n-k) = q(n)$ —— (1)

which is Non Homo. Linear Recc. Relation with const coeffs.

case I :— $q(n) = d$ (constant)

For particular solution, take $f(n) = q$

From (1)

$$q + c_1 q + c_2 q + \dots + c_k q = d$$

$$(1 + c_1 + c_2 + \dots + c_k) q = d$$

$$q = \frac{d}{1 + c_1 + c_2 + \dots + c_k} \quad ; \quad 1 + c_1 + c_2 + \dots + c_k \neq 0$$

But if $(1 + c_1 + c_2 + \dots + c_k) = 0$

then this procedure fails. Then we take $f(n) = nq$

for particular solution in (1)

$$nq + c_1(n-1)q + c_2(n-2)q + \cdots + c_k(n-k)q = d$$

If this also fails then $f(n) = n^2 q$ and so on.

Q:→ Solve the recurrence relation

$$S(n) - 2S(n-1) + 4 = 0$$

where $S(0) = 5$.

Sol:→ Given recc. relation

$$S(n) - 2S(n-1) = -4 \quad \text{——— (1)}$$

Where $S(0) = 5$.

<u>Homo. solution</u> $(S^h(n))$

Associated homo. recc. relation is

$$S(n) - 2S(n-1) = 0 \quad \text{——— (2)}$$

For char eqn, take $S(n) = a^n$ in (2)

$$a^n - 2a^{n-1} = 0$$

$$\Rightarrow a - 2 = 0$$

$$\Rightarrow a = 2$$

$$\therefore \quad S^h(n) = A(2)^n$$

<u>Particular Solution</u> $(S^p(n))$

As $q(n) = -4$ which is a constant

Take $S(n) = q$ for particular soln in (1)

$$q - 2q = -4$$

$$-q = -4$$

$$\therefore q = 4$$

$$\therefore \quad S^p(n) = 4$$

<u>Complete Solution</u>

$$S(n) = S^h(n) + S^p(n)$$

$$S(n) = S^h(n) + S^p(n)$$

$$\therefore\ S(n) = A(2)^n + 4$$

As $S(0) = 5$

$A + 4 = 5$

$A = 1$

Therefore, $S(n) = 1 \cdot (2)^n + 4 = 2^n + 4$ Ans

Q:→ solve $S(n) - 2S(n-1) + S(n-2) = 1$
with $S(0) = 2$, $S(1) = 5.5$

Sol:→ Given recc. relation

$$S(n) - 2S(n-1) + S(n-2) = 1 \quad \text{——— (1)}$$

with $S(0) = 2$, $S(1) = 5.5$

Homo Solution ($S^h(n)$)

Associated Homo recc. relation

$$S(n) - 2S(n-1) + S(n-2) = 0 \quad \text{——— (2)}$$

For char eqn, take $S(n) = a^n$ in (2)

$$a^n - 2a^{n-1} + a^{n-2} = 0$$

$$\Rightarrow a^2 - 2a + 1 = 0$$

$$\Rightarrow (a-1)^2 = 0$$

$$\Rightarrow a = 1, 1$$

$$\therefore\ S^h(n) = (A_1 + nA_2)(1)^n = A_1 + nA_2$$

Particular Solution ($S^p(n)$)

Here $g(n) = 1$ which is a constant.

For particular solution we take $S(n) = q$ in (1)

$$q - 2q + q = 1$$

$$0q = 1 \quad \text{This is the case of failure.}$$

Then we redefine $S(n) = nq$ for the particular solution.

$nq - 2(n-1)q + (n-2)q = 1$

$nq - 2nq + 2q + nq - 2q = 1$

$0q = 1$ Not possible

Again redefine $S(n) = n^2 q$ for the particular solution

$n^2 q - 2(n-1)^2 q + (n-2)^2 q = 1$

$(n^2 - 2(n-1)^2 + (n-2)^2)q = 1$

$(n^2 - 2(n^2 - 2n + 1) + (n^2 - 4n + 4))q = 1$

$(n^2 - 2n^2 + 4n - 2 + n^2 - 4n + 4)q = 1$

$2q = 1$

$\Rightarrow q = \frac{1}{2}$

$\therefore S^p(n) = \frac{n^2}{2}$

Complete solution

$S(n) = S^h(n) + S^p(n) = A_1 + nA_2 + \frac{n^2}{2}$

As $S(0) = 2$ and $S(1) = 5.5$

$A_1 = 2$

$A_1 + A_2 + \frac{1}{2} = 5.5$

$2 + A_2 + 0.5 = 5.5$

$A_2 = 3$

Therefore $S(n) = 2 + 3n + \frac{n^2}{2}$ Ans

Ex:→ Solve

(i) $a_r + 5a_{r-1} + 6a_{r-2} = f(r) = \begin{cases} 0 & r = 0, 1, 5 \\ 6 & \text{otherwise} \end{cases}$

where $a_0 = 0$, $a_1 = 1$

(ii) $S(K) - 5S(K-1) + 6S(K-2) = 2$

where $S(0) = 1$, $S(1) = -1$.

Solution :→

(i) Given reccurence relation is

$$a_r + 5a_{r-1} + 6a_{r-2} = f(r)$$

a) when $r = 0, 1, 5$; $f(r) = 0$

i.e $a_r + 5a_{r-1} + 6a_{r-2} = 0$

which is a homo recc relation with const coeffs

For char eqn take $a_r = a^r$

$$a^r + 5a^{r-1} + 6a^{r-2} = 0$$

$$\Rightarrow a^2 + 5a + 6 = 0$$

$$\Rightarrow (a+2)(a+3) = 0$$

$$\Rightarrow a = -2, -3$$

$$\therefore a_r = A(-2)^r + B(-3)^r$$

As $a_0 = 0$ and $a_1 = 1$

$A + B = 0$, $-2A - 3B = 1$

$B = -A$, $-2A + 3A = 1$

$A = 1$

$\Rightarrow B = -1$

Hence, $a_r = (-2)^r - (-3)^r$

b) If $r \neq 0, 1, 5$ then

$$a_r + 5a_{r-1} + 6a_{r-2} = 6 \qquad \text{—— } (*)$$

which is a Non Homo Recc Relation with constant coefficients.

Homo solution: (a_r^h)

Associated Homo eqn $a_r + 5a_{r-1} + 6a_{r-2} = 0$

$\therefore a_r^h = A(-2)^r + B(-3)^r$ (from (a) part)

Particular Solution:

As $f(r) = 6$ take $a_r = q$ in ⊛

$$q + 5q + 6q = 6$$
$$12q = 6$$
$$q = \frac{1}{2}$$

$$\therefore \quad a_r^p = \frac{1}{2}$$

Complete solution:

$$a_r = a_r^h + a_r^p$$
$$= A(-2)^r + B(-3)^r + \frac{1}{2}$$

As $a_0 = 0$ and $a_1 = 1$

$$A + B + \frac{1}{2} = 0 \qquad -2A - 3B + \frac{1}{2} = 1$$

$$A + B = -\frac{1}{2} \qquad -2A - 3B = \frac{1}{2}$$

$$2A + 2B = -1 \quad , \quad -4A - 6B = 1$$

on solving $B = \frac{1}{2}$, $A = -1$

$$\therefore \quad a_r = -1(-2)^r + \frac{1}{2}(-3)^r + \frac{1}{2}$$

$$= \frac{-2(-2)^r + (-3)^r + 1}{2}$$

$$= \frac{(-2)^{r+1} + (-3)^r + 1}{2} \quad \underline{\text{Ans}}$$